An Outline of the Development of the Concept of Fuzzy Multisets

Dasharath Singh, Alhaji Alkali, and Adeku Musa Ibrahim

***Abstract*—The paper argues that the classical set-theoretic foundation for mathematics is too *restrictive* to be able to model a large class of real-life problems which intrinsically involve *ambiguities*. Further, it describes how by relaxing the restrictions of *definiteness* and *distinctness* imposed on the nature of objects to form a cantorian set, the notions of *fuzzy sets* and *multisets* respectively get introduced. Finally, it explicates the relevance of generalizing fuzzy sets to fuzzy multisets.**

***Index Terms*—Classical sets, fuzzy sets, multisets, fuzzy multisets.**

I. Introduction

Essentially, mathematics studies *structures*. A structure is a collection (Bereich) of objects (events, states, etc.) with certain operations and relations defined on these objects or their n-tuples. As such, the notion of a *set* is not only fundamental to mathematics but also, it is of paramount importance in natural languages insofar as they are required to possess a structure.

Georg Ferdinand Ludbig Philip Cantor (1845–1918) formulated modern set theory. It embodies the following two basic assumptions: objects occurring in a set must be *definite* and *distinct*. That is, given a set M and an object m, either m belongs to M or it does not belong to M; if m belongs to M, it must not occur in its multiple copies. That is why, it is called a *crisp* set in contrast to other alternative possibilities such as *fuzzy sets* or *multisets*.

A crisp set in Fregean formulation [1] is a concept with a *sharp* (crisp) boundary i.e., there is no uncertainty in deciding its boundary. By implication, it may be construed that the possibility of the existence of noncrisp sets must have been envisaged by Frege.

A number of mathematicians ([2] provides some details), even during the hay-day of Cantor's intuitive set theory, expressed their discomfort with its too restrictive demand of definiteness and distinctness. It will be elaborated in the following how by relaxing the restrictions of definiteness and distinctness, the notions of fuzzy sets and multisets respectively get introduced.

It should be noted that the classical (cantorian) set theory, despite embodying severe limitations on the nature of objects to form a set, has provided a firm foundation for *bivalent* logic based mathematics in which all variables as well as relations between variables are crisp. However, if mathematics is required to model the concept of a class of objects and relations between them which may not possess crisply defined criteria, the classical paradigm based on two-valued logic is found *incapacitated* and even *inapplicable* at times. For example, *the class of all real numbers which are much larger than ten*, *the class of all handsome living statesmen across the world*, etc., cannot be characterized extensionally because their defining predicates involve *vagueness* and *uncertainty*, and admit *gradations* in values of membership beyond zero and one.

Yet, the fact remains that such imprecisely defined classes play an *important role in human thinking, particularly in the domain of pattern recognition, communication of information and abstraction* [3].

Regular *mathematics is like the traffic camera that uses a logic that has rigid limits. Sometimes, it does not work for the purposes you need, especially when something approximating human judgment is needed* [4].

The queerness of the concept of vagueness has long been drawing the attention of philosophers, logicians, and mathematicians. As noted in [5], Nietsche was the first to recognize the notion of vagueness. In course of time, various other closely related notions such as *loose concepts*, *haziness*, *borderline cases*, *fluent boundaries*, *case by grades*, etc., appeared. As mentioned earlier, Gottlob Frege [1] was the first to provide a mathematical definition of vagueness in terms of having an *unsharp boundary*. A seminal contribution towards investigating the concept of vagueness was made by [6]. The epicentre of Black's explication can be seen as a unifying thread between Bertrand Russell's and C. S. Peirce's approach.

Menger [7] argues that the notion of *probability* could adequately deal with loose concepts. He also introduced the notion of *hazy set*. However, it was not explicit until the formulation of the theory of fuzzy sets [3] that the notion of probability could not deal with vagueness and other loose concepts if the meaning of these concepts is the *absence of sharp boundaries*. Seising [8] provides a perspective account of the development of the concept of fuzziness.

A distinctive feature of the concept of *fuzziness* can be seen summarized in the following: *In contrast to the stochastic uncertainty-type vagueness, the vagueness concerning the description of the semantic meaning of events, phenomena or statements is called* fuzziness [9]. Kaushal [10] provides a good deal of illustrations to describe the relevancy of fuzzy concept in mathematics.

II. Fuzzy Sets

In view of the pervasive role played by set-theoretic foundation, it was seemingly natural to look for a set theory-

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The authors are with Mathematics Department, Ahmadu Bello University, Zaria, Nigeria (e-mail: Nigeria.mathdss@yahoo.com, alkali4real2012@gmail.com, adekubash1@gmail.com).

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like framework to model the class of problems in which the source of vagueness is not the presence of random variables rather the absence of precisely defined criteria of class membership. Fortunately, it was found forthcoming by way of relaxing the restriction of *definiteness* imposed on objects to form a Cantorian set. L. A. Zadeh was the first who formulated a set-theoretic model in [3] and titled it *fuzzy set theory* in contrast to *crisp set theory*. Fuzzy set theory is a mathematical theory to model vagueness and other loose concepts. It deals with fuzzy variables and fuzzy relations. Essentially, a fuzzy set is characterized by its elements occurring with a continuum of *degrees* (*grades*) of membership. Moreover, the grade of membership (full or partial) and non-membership (full or partial) of an element in a fuzzy set is required to be commensurate with the concept represented by the fuzzy set. A perceptive advantage can be observed in Fig. 1(a) and Fig. 1(b) in representing vague concepts such as *low*, *medium* and *high* by fuzzy sets respectively.

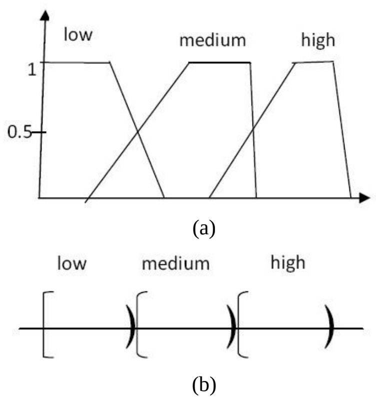
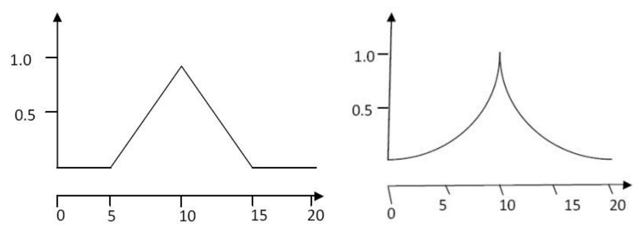


Fig. 1. A perceptive advantage.

It is instructive to note that an appropriate representation of a fuzzy set intended to model a vague concept must account for the *context* as well. For example, the concept of *high temperature* in the context of weather and in the context of nuclear reactor would require different fuzzy sets for their representations. Moreover, even for similar contexts (e.g., weather in different seasons in the same climate) representing the same concept (e.g., high temperature) distinct fuzzy sets would be required. For example, the graphs Fig. 2(a) and Fig. 2(b) represent the same concept viz., *the class of real numbers close to ten*.



(a) (b)

Fig. 2. The class of real numbers close to ten.

Some other variations in the shapes of the graph of the same concept are possible depending on the context of a particular application ([11] contains some details). In most of the applications, especially in order to achieve computational efficiency, the shapes for the membership functions of fuzzy variables are *triangular*, *bell* shaped, *trapezoidal or Gaussian functions* whereas, *right open intervals* of real numbers are used in dealing with functions of crisp variables. The triangular representation is known to be coarser than others.

Undoubtedly, the extensional representations of a vague concept by ordinary sets are mathematically sound. However, it is inherently far from reality, simply because the decision to include the points falling into a close neighborhood of each sharply defined boundary between states of a crisp variable in only one of the states involves inevitable uncertainty which is autocratically ignored. For example, crisply defined ranges to represent body temperatures of human beings viz., *not feverish*, *feverish*, and *very highly feverish*, are incongruent with *reality* which could best be brought out by some suitable nonextensional approach.

A fuzzy set is defined [3] as follows:

A *fuzzy set (class)* A in a space X of points (objects) is characterized by a *membership (characteristics) function* $f_A(x)$ which associates with each point x in X a real number in the interval $[0, 1]$, with the values of $f_A(x)$ at x representing the *grade of membership* of x in A. This approach for formulating the theory of fuzzy sets can be seen straightforwardly related to an infinite-valued logic by interpreting the membership grades as *the truth degrees* with $[0, 1]$ as its *truth degree set*. The membership function is usually denoted by $\mu_A(x)$ or just $A(x)$.

For example, let X be the real line $\mathbb{R}$ and, let A be a fuzzy set of numbers that are considerably larger than ten. Then A can be precisely, although subjectively, characterized by a membership function $\mu_A(x)$) on $\mathbb{R}$, where $\mu_A(10) = 0$, $\mu_A(15) = 0$, $\mu_A(50) = 0.01$, $\mu_A(100) = 0.05$, $\mu_A(500) = 0.5$, $\mu_A(1000) = 1$, etc., can be taken as its representative values. Customarily while listing the elements of a fuzzy set, elements with *zero degree* of membership are usually omitted.

Schematically, a fuzzy set A in X can be represented as
$\mu_A: X \longrightarrow [0, 1]$,
where μ_A is the membership function.

Also, A is a fuzzy set in X is usually represented as $A: X \longrightarrow [0, 1]$, or $A = \{(x, \mu_A(x))/x \in X\}$, a set of ordered pairs.

Equivalently, a fuzzy set is represented by a *generalized* characteristic function, first introduced in [12], in contrast to the characteristic function taking values in $\{0, 1\}$ in the case of an ordinary set. It follows that fuzzy models subsume the role of classical models and, in this sense, fuzzy set theory is a generalization of the classical set theory. As indicated in [3], other ranges of the membership function beyond the unit interval (for example, a partially ordered set) could be gainfully considered. Goguen [13] formulates a theory of the *lattice fuzzy* (*L - Fuzzy*) sets by taking a complete lattice as the range of membership function. In view of these generalizations, if the degrees of membership of a fuzzy set are real from the unit interval, it is called *standard* or *ordinary* or *first order*. It is interesting to note that the range of membership function in the case of a classical set viz., {0, 1} forms a two-element Boolean algebra, a partially ordered structure.

The notion of fuzzy sets is essentially *nonstatistical*. Although the membership function of a fuzzy set does possess some resemblance to a probability function where

X is countable (or a probability density function when X is a continuum), Probability theory is not an appropriate vehicle to deal with the kind of uncertainty that appears especially in *pattern classification* and *information processing*. This type of uncertainty seems to be more of an *ambiguity* than a statistical variation [3], [13]. Of course, probability theories can help solving problems involving stochastic- type vaguenesses.

Following the formulation of fuzzy set theory, other approaches addressing the problem of how to understand and manipulate imperfect knowledge got underway: *rough set theory* [14] and *soft set theory* [15] are two prominent directions. It is but natural that both the approaches share a lot many commonalities with the fuzzy set theory, yet being different. As Pawlak notes, the most successful one is, no doubt, is the fuzzy set theory.

During the last three decades or so, many formulations and interpretations of the theories of fuzzy sets have appeared. Infinite valued logic based approaches, *set theocratic* approaches, *model theoretic* approaches and *category theoretic* approaches have been the major directions for developing theories of fuzzy sets ([15] is a crisp and excellent survey).

As to the limitations of fuzzy set theory, the only difficulty, especially raised by soft set theoreticians, is the nature of membership function being extremely individual. On our view, it is actually the most competitive feature that makes fuzzy set theory more comprehensive and expressive than other approaches known so far to deal with vagueness and uncertainty as closely as possible to human thinking. In fact, the subjectivity involved is not bald. It is guided by taking account of both the *concept* and *context* in a harmonious framework. However, constructing appropriate membership function for a wide variety of concepts in various contexts is quite an involved task.

III. Multisets

Similar to the concept of fuzziness, the concept of *multiple-membership collections* has a long and tortuous history. Knuth [17] notes that despite frequent occurrences of multi-set like structures in mathematics, there is currently no structured way to deal with multisets. The term *multiset*, as Knuth notes, was first suggested by N.G. De Bruijn in a private communication to him. Owing to its aptness, it has replaced a variety of terms viz., *heap*, *bunch*, *bag*, *sample*, *weighted set*, *occurrence set*, *fire set*, etc.

In fact, prior to coinage of the term *multiset*, the term *bag* was in frequent use. Currently, multiset and bag are being used interchangeably. Knuth [17], along with providing a concise historical perspective of the concept of multiset, presents many results on multiset operations and their applications in mathematics, computer science, and other sciences.

The challenging task of formulating sufficiently rich mathematics of multisets started receiving serious attention from beginning of the 1970s. An updated exposition on both historical and mathematical perspective of the development of the theory of multisets can be found in [18], [19]. The underlying idea was to develop a generalization of the ordinary set theory by way of relaxing the restriction of *distinctness* on the nature of the objects forming a set. As mentioned earlier, this demand of cantorian set theory did not go hand in hand with the nature of a large number of problems arising in mathematics as well as in other *hard* and *soft* science. For example, consideration of repeated roots of a polynomial equation, repeated observations in a statistical sample, repeated hydrogen atoms in a water molecule H_2O, etc., do play a significant role. The *principle of indistinguishability* attributed to Henri Poincare [8] and elaborately described by Parker–Rhodes [19] contemplates that objects may be *identical*, *distinct* or *twins*. For example in H_2O, the two hydrogen atoms H^1 and H^2 are the same but separate, H^1 and O (or H^2 and O) are obviously distinct, while H^1 and H^1 (or H^2 and H^2) are coinciding and identical.

A *multiset* (mset for short) or a *bag* is an unordered collection of objects in which, unlike an ordinary set, objects are allowed to repeat. Each individual occurrence of an object in an mset is called its *element*. All duplicates of an object in an mset are treated indisguishables . The objects of an mset are its distinguishable (distinct) elements. The number of occurrences of an object, which is usually finite, in an mset is called its cardinality, denoted by $m_A(x)$ or $C_A(x)$ or $A(x)$. The cardinality of an mset A is the sum of the multiplicities of all its objects, denoted by $C(A)$or $|A|$. That is, $C(A) = \sum C_A(x)$ for all x in A. The *root* or *support* or *carrier* of an mset A, denoted by A^*, is the set containing all distinct elements of A. It follows that every must has a unique root set. The cardinality of the root set of an mset A is called its *dimension*.

An mset is usually represented by using square brackets, instead of curly brackets, to distinguish it from set representations. For example an mset containing one occurrence of a, two occurrences of b, and three occurrences of c is notated $[a, b, b, c, c, c]$ or $[a, b, c]_{1,2,3}$ or $[1/a, 2/b, 3/c]$ or $[a^1, b^2, c^3]$ or $[a^1 b^2 c^3]$, etc. For convenience, the curly brackets are often used if no confusion arises. There are many other forms of multiset representations. The representation of an mset in a function form, which is found quite useful especially in developing axiomatic foundations and applications in computer science, is as follows:

Let D be a domain set (universe) and let T be a numeric set. Then, a map $\alpha : D \to T$ is called

$$\begin{cases} \text{a set, if } T = \{0, 1\}; \\ \text{a multiset, if } T = \aleph \text{, the set of natural numbers including zero;} \\ \text{a signed multiset (or, hybrid/shadow set) if } T = Z\text{, the set of all integers; and} \\ \text{a fuzzy (or hazy) set if } T = [0, 1]. \end{cases}$$

Multisets have found many applications in mathematics, computer science (especially in database theory, π–calculus, membrane computing etc.), Linguistics, economics, etc. [18]-[21], are some references which contain most of the details regarding mathematics of multisets and their applications.

IV. Fuzzy Multisets

Relatively recently, in view of the aforesaid two very dominant generalizations of the ordinary set theory, efforts to develop mathematical structures characterizing classes of more complex objects, possessing both fuzziness and multiplicity, have been made. The concept in view is that of a *fuzzy multiset*. In order to distinguish it, an ordinary nonfuzzy multiset is called a *crisp (pure) multiset*. For example, $A = \{(x,1),(x,0.4),(x,0.3),(y,0.8),(y,0.6)\}$, equivalently represented as $A = \{\{1,0.4,0.3\}/x, \{0.8,0.6\}/y\}$, is a fuzzy multiset in $X = \{x,y,z\}$.

Fuzzy multisets have been introduced in [22], and further studied in [23]-[25] and others. As Yager uses the term *bag* for *multiset*, similarly he uses the term *fuzzy bag* for *fuzzy multiset.* A typical approach for generalizing multiset to fuzzy multisets is to fuzzify the multiplicities of objects of a multiset. Formally, a fuzzy multiset in some universe set X is a multiset in $X \times [0,1]$. That is, a fuzzy multiset is a multiset of pairs, where the first part of each pair is an element of X and the second part is the degree to which the first part belongs to fuzzy multiset.

Besides, possibilities of generalizing fuzzy multisets are currently being investigated [25], [26]. For example, a generalized fuzzy multiset is obtained if several objects of X appear with the same grades in the fuzzy multiset considered in the example above.

Syropoulos [26] formulates *L - Fuzzy hybrid sets* on fuzzyfying the objects of a hybrid multiset. It should be noted that hybrid sets have found numerous applications in mathematics and computer science. For example a hybrid set can be used for describing *roots* and *poles* of a rational function [27].

In view of overriding influence of fuzzy set theory, generalizations such as *rough fuzzy sets* and *fuzzy rough sets* [28], *fuzzy soft sets* [29], etc., have also been undertaken. Future studies include further generalizations of the theory of fuzzy multisets, especially pertaining to their applications.

Round numbers are always false
Samuel Johnson [c.1750].

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Dasharath Singh is a former professor (IIT Bombey) and currently professor, Department of Mathematics, Ahmadu Bello University, Zaria, Nigeria. Areas of specialization are Sets, Multisets, Fuzzy set, Fuzzy multisets and soft sets theory

Alhaji Jibril Alkali is a lecturer and Ph. D scholar in Department of Mathematics Ahmadu Bello University, Zaria, Nigeria, and Specializing in Fuzzy Multisets.

Adeku Musa Ibrahim is a Senior lecturer, currently in Department of mathematics, Ahmadu Bello University, Zaria, Nigeria, and Specializing in Multiset Algebras.